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EHRENFEST THEOREM IN PRECANONICAL QUANTIZATION

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Abstract. We discuss the precanonical quantization of fields which is based on the De Donder–Weyl (DW) Hamiltonian formulation and treats the space and time variables on an equal footing. Classical field equations in DW Hamiltonian form are derived as the equations for the expectation values of precanonical quantum operators. This field-theoretic generalization of the Ehrenfest theorem demonstrates the consistency of three aspects of precanonical field quantization: (i) the precanonical representation of operators in terms of the Clifford (Dirac) algebra valued partial differential operators, (ii) the Dirac-like precanonical generalization of the Schrödinger equation without the distinguished time dimension, and (iii) the definition of the scalar product for calculation of expectation values of operators using the precanonical wave functions.

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I am very honoured to contribute a paper to the volume dedicated to Professor Jan Sławianowski. I deeply appreciate his encouraging support during my hard years in Warsaw in the second half of the 1990s. Some aspects of the Ehrenfest theorem in (what I later called) precanonical quantization of fields were discussed with him at his Laboratory of Analytical Mechanics and Field Theory already around 1997. Moreover, one of my earlier attempts to understand a covariant field quantization leading in the classical limit to the generalized Hamilton-Jacobi theories in the calculus of variations [24, 30] was inspired by the geometric discussion of the van Vleck determinant in Sławianowski's monumental book on the geometry of phase space [32].

1. Introduction

The canonical Hamiltonian formalism in field theory is not the only possible extension of the Hamiltonian formalism from mechanics to field theories described by multiple integral variational problems (see e.g. [24, 30]). Moreover, the alternative extensions, such as the De Donder–Weyl (DW) theory [3, 35], actually do not need to distinguish a time dimension and, therefore, are not restricted to the globally hyperbolic space-times. It is natural to ask if the alternative Hamiltonian formulations can lead to a certain reformulation of the quantization procedure in field theory, which would be more general than the canonical quantization. Though the DW theory has been known in the calculus of variations since the 1930s, it is the lack of a suitable generalization of the Poisson bracket to this framework which made it impossible to use for field quantization. When such a generalization was found in 1993 [9, 14, 15], it has paved the way to the approach to field quantization based on the DW theory, which I later called *precanonical quantization*. The term reflects the nature of mathematical structures of the DW theory, which are in a sense intermediate between the Lagrangian formalism and the canonical Hamiltonian formalism.

The Ehrenfest theorem initially has been playing an important heuristic role in developing a field quantization based on the DW Hamiltonian formulation in field theory. However, the importance of this role is probably not obvious from the papers which I have published at different stages of the development of the theory [10–13]. In this paper I would like to present a more systematic treatment of the Ehrenfest theorem in the quantum theory of fields which is based on precanonical quantization. A more naive treatment, which is found in my earlier papers, is now improved by a proper definition of the scalar product of Clifford-valued precanonical wave functions and a modified notion of self-adjoint operators with respect to this scalar product, which comply with the fact that a quantum formalism

resulting from precanonical quantization is essentially the one with an indefinite metric Hilbert space.

Note that the ability of precanonical quantization to reproduce the correct classical field equations on the average can be considered as a test of precanonical representation of operators, the precanonical analogue of the Schrödinger equation and the prescription for the calculation of expectation values of operators using the Clifford algebra valued precanonical wave functions.

We proceed as follows. In Section 2 we discuss the precanonical quantization starting from the outline of the DW Hamilonian formulation and the Poisson-Gerstenhaber brackets of differential forms, which generalize the Poisson brackets to the DW theory. The quantization based on these brackets is outlined in Section 2.3. In Section 2.4 we briefly discuss a connection between the precanonical field quantization and the functional Schrödinger representation in QFT. Different aspects of the Ehrenfest theorem in the context of precanonical field quantization are discussed in Sections 3–5. We consider the Ehrenfest theorem in the case of interacting scalar fields in flat space-time in Section 3, pure Yang-Mills theory in Section 4, and the scalar fields in curved space-time in Section 5. The latter consideration allows us to identify the connection term in the curved space-time generalization of the precanonical Schrödinger equation with the spin-connection. The concluding remarks are found in Section 6.

2. Precanonical Field Quantization

Let us first outline the basic elements of precanonical quantization. Instead of using the canonical Hamiltonian formalism, which requires a decomposition into the space and time, we start from the De Donder–Weyl extension of the Hamiltonian formulation of the Euler-Lagrange equations to field theory [24, 30], where no distinction between the space and time variables is required.

2.1. De Donder–Weyl Hamiltonian Formulation

Let us consider a field theory given by a Lagrangian density $L = L(y^a, y^a_\mu, x^\nu)$, which is a function of the space-time variables x^μ, field variables y^a and the coordinates of their first space-time derivatives (first jets) y^a_μ, such that on a specific field configuration $y^a = y^a(x)$, $y^a_\mu = \partial_\mu y^a(x)$. We can define new Hamiltonian-like variables without the distinction between the space and time variables: the *polymomenta*

$$p^\mu_a := \frac{\partial L}{\partial y^a_\mu} \tag{1}$$

and the *DW Hamiltonian function*

$$H(y^a, p^\mu_a, x^\mu) := y^a_\mu(y,p)p^\mu_a - L. \tag{2}$$

Then, if the DW Legendre transformation $(y^a, y^a_\mu) \to (y^a, p^\mu_a)$ is regular, i.e.,

$$\det||\partial^2 L/\partial y^a_\mu \partial y^b_\nu|| \neq 0 \tag{3}$$

the Euler-Lagrange field equation can be written in *DW Hamiltonian form*

$$\partial_\mu y^a(x) = \frac{\partial H}{\partial p^\mu_a}, \quad \partial_\mu p^\mu_a(x) = -\frac{\partial H}{\partial y^a}. \tag{4a, b}$$

In what follows we denote $\frac{\partial}{\partial y^a}$ as ∂_a.

Note that it is also possible to construct an analogue of the Hamilton-Jacobi (HJ) theory associated with the DW Hamiltonian formulation. The corresponding DWHJ equation [24, 30, 35]

$$\partial_\mu S^\mu + H\left(y^a, p^\mu_a = \partial_a S^\mu, x^\mu\right) = 0 \tag{5}$$

defines the solutions of field equations in terms of the wave fronts corresponding to the eikonal functions $S^\mu(y^a, x^\mu)$ on the finite dimensional analogue of the configuration space, i.e. the space of field variables y^a and space-time variables x^μ. The very existence of such a Hamilton-Jacobi theory on the finite dimensional space of y^a and x^μ rises the question about the existence of a formulation of quantum field theory in terms of the wave functions on this space, which leads to the DWHJ equation in the classical limit.

2.1.1. Example: Classical Interacting Scalar Fields

In the case of the theory of interacting scalar fields y^a with the Lagrangian

$$L = \frac{1}{2}\partial_\mu y^a \partial^\mu y_a - V(y) \tag{6}$$

where $V(y)$ includes both the mass terms like $\frac{1}{2}\frac{m^2}{\hbar^2}y^2$ and the interactions, we obtain $p^\mu_a = \partial^\mu y_a$ and

$$H = \frac{1}{2}p^\mu_a p^a_\mu + V(y). \tag{7}$$

The DW Hamiltonian equations obtained from (4)

$$\partial_\mu p^\mu_a = -\partial_a V, \quad \partial_\mu y^a = p^a_\mu \tag{8}$$

are just the first order form of the coupled nonlinear Klein-Gordon equations for scalar fields $y^a = y^a(x)$.

The DWHJ equation (5) for interacting scalar fields takes the form of a partial differential equation

$$\partial_\mu S^\mu + \frac{1}{2}\frac{\partial S^\mu}{\partial y^a}\frac{\partial S_\mu}{\partial y_a} + V(y) = 0 \tag{9}$$

where $S^\mu(y^a, x^\mu)$ are eikonal functions on the finite dimensional covariant configuration space. By treating the space $\mathbf{x}$ and time $t := x^0$ variables differently and constructing a functional

$$\mathbf{S}([y^a(\mathbf{x})], t) := \int \mathrm{d}\mathbf{x}\, S^0(y^a = y^a(\mathbf{x}), \mathbf{x}, t)$$

we can show [17] that, as a consequence of the DWHJ equation (9), the functional $\mathbf{S}$ obeys the standard Hamilton-Jacobi equation in functional derivatives, which is familiar from the canonical Hamiltonian formalism

$$\partial_t \mathbf{S} + \int dy\, \Big(\frac{1}{2}\frac{\delta \mathbf{S}}{\delta y^a(\mathbf{x})}\frac{\delta \mathbf{S}}{\delta y_a(\mathbf{x})} + \frac{1}{2}\big(\nabla y^a(\mathbf{x})\big)^2 + V\big(y(\mathbf{x})\big)\Big) = 0.$$

This is one of the examples of how the DW (precanonical) Hamiltonian structures *pre*cede the canonical ones.

2.2. Poisson Brackets in DW Hamiltonian Formulation

Quantization based on the DW Hamiltonian-like framework requires a suitable generalization of Poisson brackets. We found a generalization of the geometric construction of Poisson brackets in analytical mechanics (see e.g. [32]) to the DW Hamiltonian framework, where it is based on a higher degree generalization of the symplectic structure to the extended *polymomentum phase space* of variables $z^M := (y^a, p^\mu_a, x^\mu)$. Namely, this generalization is given by the *polysymplectic form*[1] [9, 14]

$$\Omega = \mathrm{d}p^\mu_a \wedge \mathrm{d}y^a \wedge \varpi_\mu \tag{10}$$

where $\varpi_\mu := \partial_\mu \rfloor \varpi$ and $\varpi := \mathrm{d}x^1 \wedge .. \wedge \mathrm{d}x^n$. Thus, in field theory on n-dimensional space-time a generalization of the symplectic form is a form of degree $(n+1)$. The particular form of (10) follows from the Poincaré-Cartan (PC) form corresponding to the DW theory [5] and the geometric representation of solutions of classical field equations in terms of multivector fields on the polymomentum

[1] This object can be defined as a representative of a certain equivalence class of forms, see [14]. For the related discussions see also [4, 26, 27, 29].

phase space (see [9, 14] for details). Namely, the DW Hamiltonian equations can be represented as the equations of the integral surfaces of n-multivector fields $\overset{n}{X}$, such as [9, 14]

$$\overset{n}{X} \lrcorner\, \Omega = (-)^n \mathrm{d}H. \tag{11}$$

Thinking about the introduction of a Poisson bracket, we conclude that the map between 0-forms and n-multivectors in (11) should be generalized to include the horizontal (semi-basic) forms of other degrees

$$\overset{n-p}{X} \lrcorner\, \Omega = \mathrm{d}\overset{p}{F}, \quad p = 0, 1, ..., (n-1) \tag{12}$$

where $\overset{p}{F} := \frac{1}{p!} F_{\mu_1 \ldots \mu_p}(y^a, p^\nu_a, x^\nu)\, \mathrm{d}x^{\mu_1} \wedge ... \wedge \mathrm{d}x^{\mu_1}$. This map is also suggested by the *polysymplectomorphism* symmetry introduced in [9] in terms of the Lie derivatives with respect to the multivector fields. Note that the map in (12) exists only for a special class of forms called *Hamiltonian forms* in [9, 14] (see also [15] for an explicit formula for the Hamiltonian forms) and it maps those forms to the equivalence classes of multivector fields modulo the annihilators of Ω: $\overset{p}{X} \lrcorner\, \Omega = 0, \quad p = 2, ..., n$.

The above constructions lead to the following formula for the Poisson bracket of two Hamiltonian forms $\overset{p}{F}_1$ and $\overset{q}{F}_2$

$$\{\![\overset{p}{F}_1, \overset{q}{F}_2]\!\} = (-)^{(n-p)} \overset{n-p}{X}_1 \lrcorner\, \mathrm{d}\overset{q}{F}_2 \tag{13}$$

which gives rise to the graded Lie algebra structure on Hamiltonian forms, where the grade of a p-form with respect to the bracket operation is $(n-p-1)$. It is easy to see that the bracket of p and q forms is a Hamiltonian form of degree $(p+q-n+1)$.

If we want a true Poisson bracket, we also need the bracket to obey an analogue of the Leibniz rule. From the definition of Hamiltonian forms in (12) it follows that Hamiltonian p-form is poly-linear of degree $(n-p)$ in polymomenta [15]. Therefore, the exterior product of two Hamiltonian forms is not a Hamiltonian form in general. Nevertheless, we found the product operation with respect to which the space of Hamiltonian forms is closed. It is called the *co-exterior product* [15] and denoted as $\bullet$

$$\overset{p}{F} \bullet \overset{q}{F} := *^{-1}(*\overset{p}{F} \wedge *\overset{q}{F}) \tag{14}$$

where $*$ is the Hodge duality operator on the space-time. This product requires only a volume n-form on the space-time for its definition [13].

Thus we see that a p-form has the grade $(n-p)$ with respect to the $\bullet$-product, which is different by one from its degree with respect to the bracket operation $\{\![\cdot,\cdot]\!\}$. We can also check that the bracket in (13) is a graded derivation with respect to the co-exterior product, i.e. the graded Leibniz rule is fulfilled by the graded Lie bracket with respect to the graded commutative product $\bullet$. Therefore, the space of Hamiltonian forms with the operations $\{\![\cdot,\cdot]\!\}$ and $\bullet$ is the *Gerstenhaber algebra* [14,15]. This structure generalizes the Poisson algebra structure to field theory within the DW Hamiltonian formulation. In this formulation the dynamical variables are represented by the Hamiltonian forms on the polymomentum phase space.

A connection between the Poisson-Gerstenhaber brackets on forms in the DW theory and the standard Poisson brackets in the canonical Hamiltonian formalism, which are defined on the functionals of field configurations in the canonical phase space, has been discussed in [7, 14, 33].

The bracket defined in (13) allows us to calculate simple brackets between the Hamiltonian forms constructed from the field and polymomenta variables, which will generalize the canonical brackets, viz.

$$\{\![p_a^\mu \varpi_\mu, y^b]\!\} = \delta_a^b, \ \{\![p_a^\mu \varpi_\mu, y^b \varpi_\nu]\!\} = \delta_a^b \varpi_\nu, \ \{\![p_a^\mu, y^b \varpi_\nu]\!\} = \delta_a^b \delta_\nu^\mu. \quad (15a,b,c)$$

Moreover, the Poisson-Gerstenhaber bracket in (13) allows us to write the equations of motion of Hamiltonian $(n-1)$-forms $F := F^\mu(y^a, p_a^\mu, x)\varpi_\mu$ in terms of the bracket with the DW Hamiltonian function H. In n-dimensional Minkowski space

$$\mathbf{d}\bullet F = (-1)^n \{\![H, F]\!\} + \mathrm{d}^h \bullet F \tag{16}$$

where $\mathbf{d}\bullet$ denotes the *total co-exterior differential* of a p-form $\overset{p}{F}$

$$\mathbf{d}\bullet \overset{p}{F} := \frac{1}{(n-p)!} \frac{\partial}{\partial z^M} F^{\mu_1 \ldots \mu_{n-p}}\, \partial_\mu z^M(x) \mathrm{d}x^\mu \bullet \varpi_{\mu_1 \ldots \mu_{n-p}} \tag{17}$$

$\varpi_{\mu_1 \ldots \mu_{n-p}} := \partial_{\mu_1 \ldots \mu_{n-p}} \rfloor \varpi$, and d^h is the *horizontal co-exterior differential*

$$\mathrm{d}^h \bullet \overset{p}{F} := \frac{1}{(n-p)!} \partial_\mu F^{\mu_1 \ldots \mu_{n-p}} \mathrm{d}x^\mu \bullet \varpi_{\mu_1 \ldots \mu_{n-p}}. \tag{18}$$

By substituting the $(n-1)$-form variables from the fundamental brackets (15) into (16) we reproduce the DW Hamiltonian equations (4). Note that equation (16) generalises the Poisson bracket form of the equations of motion of a function on the phase space $F(q,p,t)$ in mechanics: $\frac{\mathrm{d}}{\mathrm{d}t} F = \{H, F\} + \partial_t F$.

2.3. Precanonical Quantization

Precanonical quantization is based on a generalization of the Dirac rule of canonical quantization, which relates the Poisson brackets with the commutators of quantum operators, to the Poisson-Gerstenhaber brackets in the DW theory

$$[\hat{A}, \hat{B}] = -\mathrm{i}\hbar \widehat{\{\![A, B]\!\}}. \tag{19}$$

The mathematical and physical reasons of why the Dirac quantization rule allows us to obtain a quantum description from the classical one, though not uniquely, is a separate great issue, which we have very little to say about. Here we take it as a technical postulate of quantum theory.

Let us quantize the fundamental precanonical brackets in (15) (see [11, 12]). In the y-representation, when y^b are multiplicative operators, from quantization of (15a) we obviously obtain

$$\widehat{p_a^\nu \varpi_\nu} = -\mathrm{i}\hbar \partial_a \tag{20}$$

i.e. a classical $(n-1)$-form is represented by a quantum operator of form degree 0. This representation is also consistent with quantization of (15b), which, however, does not specify the operator of the form $\widehat{\varpi}_\nu$. Quantization of (15c) leads to the commutator

$$[\hat{p}_a^\mu, y^b \widehat{\varpi}_\nu] = \hat{p}_a^\mu \circ y^b \widehat{\varpi}_\nu - y^b \widehat{\varpi}_\nu \circ \hat{p}_a^\mu = \mathrm{i}\hbar \delta_a^b \delta_\nu^\mu \tag{21}$$

where $\circ$ denotes a composition law of operators. Therefore, $\hat{p}_a^\mu = \mathrm{i}\hbar \hat{\epsilon}^\mu \otimes \partial_a$ and

$$\hat{\epsilon}^\mu \circ \widehat{\varpi}_\nu = \delta_\nu^\mu, \qquad \hat{\epsilon}^\mu \circ \widehat{\varpi}_\nu - \widehat{\varpi}_\nu \circ \hat{\epsilon}^\mu = 0. \tag{22}$$

It is easy to see that these relations can be fulfilled if $\hat{\epsilon}^\mu$ and $\widehat{\varpi}_\nu$ are represented by Dirac matrices and $\circ$ is their symmetric product, i.e.,

$$\widehat{\varpi}_\nu = \frac{1}{\varkappa}\gamma_\nu, \qquad \hat{\epsilon}^\mu = \varkappa \gamma^\mu \tag{23}$$

where $\frac{1}{\varkappa}$ is a small constant of the dimension of $(n-1)$-volume, which appears on the purely dimensional grounds. Therefore, the polymomenta are represented by the Clifford algebra valued operators

$$\hat{p}_a^\mu = -\mathrm{i}\hbar \varkappa \gamma^\mu \partial_a. \tag{24}$$

The bracket form of field equations in (16) allows us to guess the form of *precanonical Schrödinger equation*

$$\mathrm{i}\hbar \varkappa \gamma^\mu \partial_\mu \Psi = \widehat{H} \Psi \tag{25}$$

where the precanononical wave function Ψ is a Clifford-valued wave function on the finite dimensional covariant configuration space: $\Psi(y^a, x^\mu)$. In the following sections we will see that this form of the Schrödinger equation is consistent with the Ehrenfest theorem.

Note that the Dirac operator in the left hand side of (25) is a quantum version of $(-)^{n-1}\mathrm{d}\bullet$, which is generated by the (commutator related to) the bracket with H in (16). Hence, we can identify the quantum operator of $\mathrm{d}x^\mu\bullet$ with $(-)^{n-1}\varkappa\gamma^\mu$. This observation will be used later in the calculation in equation (44).

2.3.1. Example: Quantum Interacting Scalar Fields

We can obtain an explicit expression of the operator of the DW Hamiltonian for the system of interacting scalar fields (7) by calculating the bracket

$$\{\![p^\mu_a p^a_\mu, y^b\varpi_\nu]\!\} = 2p^b_\nu \tag{26}$$

and quantizing it using the already known representation of $\hat{p}^\mu_a$ and $\hat{\varpi}_\nu$. The result is [10–12]

$$\hat{H} = -\frac{1}{2}\,\hbar^2\varkappa^2\frac{\partial^2}{\partial y^a\partial y_a} + V(y). \tag{27}$$

For the free scalar field $V(y) \sim m^2y^2$, so that $\hat{H}$ represents a harmonic oscillator in the space of field variables y. This theory can be easily solved and the precanonical wave functions can be written down explicitly (see e.g. [12, 16]).

2.4. Precanonical Quantization and Standard QFT

The functional Schrödinger representation is one of the standard descrptions of quantum fields, though not the most widely used one. There is an excellent textbook by Hatfield [6], which treats many standard aspects of QFT using the functional Schrödinger representation. In this picture the states of quantum fields are described by the Schrödinger wave functionals $\mathbf{\Psi}([y^a(\mathbf{x})], t)$, which are functionals of field configurations $y^a(\mathbf{x})$ at a given instant of time t (we use the notation $x^\mu := (\mathbf{x}, t)$).

It is natural to ask how this description is related to the description in terms of precanonical wave functions $\Psi(y^a, x^\mu)$. A comparison of the probabilistic interpretations of the Schrödinger wave functional $\mathbf{\Psi}([y^a(\mathbf{x})], t)$ (an amplitude of finding a field configuration $y^a(\mathbf{x})$ at the instant t) and the precanonical wave function $\Psi(y^a, x^\mu)$ (an amplitude of finding a value of the field y^a at the space-time point x^μ) suggests that the former can be represented as a combination of the latter taken

along a specific configuration $y^a = y^a(\mathbf{x})$. This idea has been explored in several papers [10,17–19] and it has resulted in the following formula, which expresses the Schrödinger wave functional in terms of the Volterra's multidimensional *product integral* [31, 34] of precanonical wave functions restricted to the surface Σ in the space of (y^a, x^μ), which represents the field configuration $y = y(\mathbf{x})$ at the instant of time t

$$\mathbf{\Psi}([y(\mathbf{x})], t) = \mathrm{Tr}\left\{ \prod_{\mathbf{x}} e^{-\mathrm{i} y(\mathbf{x})\alpha^i \partial_i y(\mathbf{x}) \mathrm{d}\mathbf{x}} \Psi_\Sigma(y(\mathbf{x}), \mathbf{x}, t)_{\big|\frac{1}{\varkappa}\beta \mapsto \mathrm{d}\mathbf{x}} \right\}. \tag{28}$$

Here the notation $\Psi_{|\frac{1}{\varkappa}\beta \mapsto \mathrm{d}\mathbf{x}}$ means that every $\beta/\varkappa$ in the expression of Ψ is replaced by $\mathrm{d}\mathbf{x}$ before the product integral is evaluated. In [18, 19] it is shown that the canonical functional derivative Schrödinger equation for $\mathbf{\Psi}([y(\mathbf{x})], t)$ can be derived from the precanonical Schrödinger equation (25) in the vanishing $1/\varkappa$ limit or, more precisely, in the singular limit when $\beta\varkappa$ is mapped to $\delta^{n-1}(\mathbf{0})$. Formula (28) is a consequence of this derivation. In [19] it has been explicitly demonstrated how equation (28) allows us to construct the well known expression of the vacuum state wave functional of the free scalar field [6] from the ground state solution of the precanonical Schrödinger equation for the free scalar field.

The conclusion from those considerations is that the standard QFT obtained from the canonical quantization is a limiting case corresponding to an infinitesimal $\frac{1}{\varkappa} \to 0$ of the description of quantum fields obtained from the precanonical quantization.

3. Ehrenfest Theorem

There has been some uncertainty regarding the nature of the wave function in precanonical quantization. In my earlier papers [10–12] I was tending to assume that the precanonical wave function $\Psi(y, x)$ is spinor-valued rather than Clifford algebra valued. One of the reasons was that the analogue of the Ehrenfest theorem was most straightforwadly provable with the spinor-valued wave functions. Besides, the positive definiteness of $\overline{\Psi}\gamma^0\Psi$ for Dirac spinors, and the corresponding conservation law, which was following from the Dirac-like precanonical Schrödinger equation (25), seemed to be a guarantee that the theory does have a meaningful probabilistic interpretation, in spite of the fact that the prescription of the calculation of expectation values of operators was based essentially on the scalar $\overline{\Psi}\Psi$, which is not positive definite and even not preserved under the space-time translations. Such a dichotomy of inner products is typical for the theories with an indefinite metric Hilbert space. Thus the principal advantage of preferring the Dirac spinor wave functions over the Clifford algebra valued wave functions seems to

disappear and we have to take seriously into account the fact that the quantum formalism which follows from precanonical quantization is the one with an indefinite metric Hilbert space.

In a later work on the relation of precanonical wave functions with the Schrödinger wave functional [18, 19] we have seen that the constructions most naturally work for matrix-valued (i.e., the space-time Clifford (Dirac) algebra valued) precanonical Ψ-s, rather than for spinor-valued ones, i.e. valued in the minimal ideals of the Clifford algebra.

The treatment of the Ehrenfest theorem in this paper is different from our previous more naive considerations in that the precanonical wave function is taken to be Clifford algebra valued, and the definitions of the scalar product and the notion of self-adjointness of operators is consistent with the constructions known from the theories of the indefinite metric Hilbert spaces, with $\beta = \gamma^0$ playing the role of the so-called J-metric [1].

If the wave function is a spinor Ψ, its conjugate is $\overline{\Psi} := \Psi^\dagger\beta$. However, for a general Clifford-valued wave function the conjugate one is defined as $\overline{\Psi} := \beta\Psi^\dagger\beta$. By taking the Hermite conjugate of the precanonical Schrödinger equation (25) and multiplying it from the left and right by β, and assuming that the operator $\hat{H}$ is generalized self-adjoint in the sense that $\beta\hat{H}^\dagger\beta = H$, we can write the equation of $\overline{\Psi}$ in the form

$$\mathrm{i}\hbar\varkappa\partial_\mu\overline{\Psi}\gamma^\mu = -\hat{H}\overline{\Psi} \tag{29}$$

where we have also used the property $\beta\gamma^{\dagger\mu}\beta = \gamma^\mu$.

Now we can prove the conservation law

$$\partial_\mu \int \mathrm{d}y\ \mathrm{Tr}\big(\overline{\Psi}\gamma^\mu\Psi\big) = 0 \tag{30}$$

where $\mathrm{d}y := \prod_a \mathrm{d}y^a$.

Indeed (for simplicity, we assume henceforth in calculations that $\hbar = 1, \varkappa = 1$)

$$\begin{aligned}\mathrm{i}\partial_\mu \int \mathrm{d}y\ \mathrm{Tr}\big(\overline{\Psi}\gamma^\mu\Psi\big) &= \int \mathrm{d}y\ \mathrm{Tr}\big(\mathrm{i}\partial_\mu\overline{\Psi}\gamma^\mu\Psi + \overline{\Psi}\gamma^\mu\mathrm{i}\partial_\mu\Psi\big)\\ &= \int \mathrm{d}y\ \mathrm{Tr}\big(-\hat{H}\overline{\Psi}\Psi + \overline{\Psi}\hat{H}\Psi\big) = 0.\end{aligned} \tag{31}$$

Similarly, we can obtain

$$
\begin{aligned}
\mathrm{i}\partial_\mu \int \mathrm{d}y\ \mathrm{Tr}\big(\overline{\Psi}\gamma^\mu\partial_a\Psi\big) &= \int \mathrm{d}y\ \mathrm{Tr}\big(\mathrm{i}\partial_\mu\overline{\Psi}\gamma^\mu\partial_a\Psi + \overline{\Psi}\gamma^\mu\mathrm{i}\partial_\mu\partial_a\Psi\big) \\
&= \int \mathrm{d}y\ \mathrm{Tr}\big(-\hat{H}\overline{\Psi}\partial_a\Psi + \overline{\Psi}\partial_a\circ\hat{H}\Psi\big) \\
&= \int \mathrm{d}y\ \mathrm{Tr}\big(-\overline{\Psi}\hat{H}\circ\partial_a\Psi + \overline{\Psi}\partial_a\circ\hat{H}\Psi\big) \\
&= \int \mathrm{d}y\ \mathrm{Tr}\big(\overline{\Psi}(\partial_a\hat{H})\Psi\big) = \langle\partial_a\hat{H}\rangle.
\end{aligned}
\tag{32}
$$

Taking into account the precanonical representation of the operator of polymomenta (24) this result shows that the second DW Hamiltonian equation (4b) is fulfilled on the average

$$
\partial_\mu\langle\hat{p}_a^\mu\rangle = -\langle\partial_a\hat{H}\rangle
\tag{33}
$$

if the following prescription for the calculation of expectation values of precanonical operators is adopted

$$
\langle\hat{O}\rangle(x) = \int \mathrm{d}y\ \mathrm{Tr}\Big(\overline{\Psi}(y,x)\hat{O}\Psi(y,x)\Big).
\tag{34}
$$

Note that the right hand side of (33) can be understood as follows:

$$
-\langle\partial_a\hat{H}\rangle = \langle[\hat{H},\partial_a]\rangle = \langle[\hat{H},\frac{\mathrm{i}}{\hbar}\widehat{p_a^\nu\varpi_\nu}]\rangle = \langle\widehat{\{\![H, p_a^\nu\varpi_\nu]\!\}}\rangle.
\tag{35}
$$

Next, let us consider

$$
\partial_\mu\langle y^a\rangle = \int \mathrm{d}y\ \mathrm{Tr}\big(\partial_\mu\overline{\Psi}y^a\Psi + \overline{\Psi}y^a\partial_\mu\Psi\big).
\tag{36}
$$

By multiplying the precanonical Schrödinger equation (25) and its conjugate (29) by γ^μ we can write

$$
\mathrm{i}\partial_\mu\Psi = \gamma_\mu\hat{H}\Psi - \mathrm{i}\gamma_{\mu\nu}\partial^\nu\Psi, \qquad \mathrm{i}\partial_\mu\overline{\Psi} = -\hat{H}\overline{\Psi}\gamma_\mu + \mathrm{i}\partial^\nu\overline{\Psi}\gamma_{\mu\nu}.
\tag{37}
$$

By substituting (37) into (36) we obtain

$$
\begin{aligned}
\mathrm{i}\partial_\mu\langle y^a\rangle = \int \mathrm{d}y\ \mathrm{Tr}\Big(&\big(-\hat{H}\overline{\Psi}\gamma_\mu + \mathrm{i}\partial^\nu\overline{\Psi}\gamma_{\mu\nu}\big)y^a\Psi \\
&+ \overline{\Psi}y^a\big(\gamma_\mu\hat{H}\Psi - \mathrm{i}\gamma_{\mu\nu}\partial^\nu\Psi\big)\Big) \\
= \int \mathrm{d}y\ \mathrm{Tr}\Big(&\overline{\Psi}\big([y^a\gamma_\mu,\hat{H}] - \mathrm{i}y^a\gamma_{\mu\nu}\overleftrightarrow{\partial^\nu}\,\big)\Psi\Big)
\end{aligned}
\tag{38}
$$

where $a \overset{\leftrightarrow}{\partial}_\mu b := a\partial_\mu b - (\partial_\mu a)b$.

While the first term in (38) reproduces the statement of the Ehrenfest theorem for the first DW Hamiltonian equation in (4a), the nature of the second term is not clear. In fact, equations (37) are formal and their use should take into account the integrability condition $\partial_{[\mu}\partial_{\nu]}\Psi = 0$, which leads to a rather complicated system of additional equations. For this reason the use of equations (37) to prove the Ehrenfest theorem, in the way it is done in (38), does not appear to be justified.

In order to prove the Ehrenfest theorem for the first DW Hamiltonian equation in (4a) by exploiting the same mechanism as in (32), let us use the fact that, according to the precanonical fundamental bracket in (15c), the variable (precanonically) conjugate to p_a^μ is an $(n-1)$-form $y^a\varpi_\nu$, for which equation (4a) can be rewritten as

$$\partial^\mu(y^a\varpi_\mu) = \frac{\partial H}{\partial p_a^\mu}\varpi_\mu = p_a^\mu\varpi_\mu \tag{39}$$

where in the last equality we use the expression of the DW Hamiltonian for the interacting scalar fields, see (7). For the expectation value of the operator $\widehat{y^a\varpi_\nu} = \frac{1}{\varkappa}y^a\gamma_\mu$ we obtain

$$\begin{aligned}
\mathrm{i}\partial^\mu\langle\widehat{y^a\varpi_\mu}\rangle &= \mathrm{i}\partial_\mu\int \mathrm{d}y\ \mathrm{Tr}\big(\overline{\Psi}\gamma^\mu y^a\Psi\big) = \mathrm{i}\int \mathrm{d}y\ \mathrm{Tr}\big(\partial_\mu\overline{\Psi}\gamma^\mu y^a\Psi + \overline{\Psi}y^a\gamma^\mu\partial_\mu\Psi\big)\\
&= \int \mathrm{d}y\ \mathrm{Tr}\big(-\hat{H}\overline{\Psi}y^a\Psi + \overline{\Psi}y^a\hat{H}\Psi\big) \qquad\qquad (40)\\
&= \int \mathrm{d}y\ \mathrm{Tr}\big(\overline{\Psi}[y^a,\hat{H}]\Psi\big) = \int \mathrm{d}y\ \mathrm{Tr}\big(\overline{\Psi}\partial_a\Psi\big) = \mathrm{i}\langle\widehat{p_a^\mu\varpi_\mu}\rangle
\end{aligned}$$

where in the last line we use the expression of the DW operator of interacting scalar fields (27).

Thus, we have shown in (40) that the first DW Hamiltonian equation in (4a) written in the form (39) is satisfied on the average as the equation for the expectation values of the corresponding operators. Together with equation (33) it proves the Ehrenfest theorem for the precanonically quantized system of interacting scalar fields in flat space-time: the classical DW Hamiltonian equations of this system are fulfilled by the expectation values of the corresponding precanonical operators.

However, there remains certain dissatisfaction due to the fact that we were able to prove the Ehrenfest theorem only for a specific form of the DW Hamiltonian equation: namely, the one given by (39).

Looking on the proofs in equations (32) and (40), we see that the right hand sides of the DW Hamiltonian equations are reproduced as expectation values of certain

commutators with $\hat{H}$. It suggests that the Ehrenfest type statement is more naturally obtained for the Poisson bracket form of the DW Hamiltonian equations rather than for their naive form in (4).

Let us recall that in the DW Hamiltonian theory we have shown that the DW Hamiltonian equations in Minkowski space can be written in the form (cf. (16))

$$\begin{aligned} \mathbf{d} \bullet p_a^\mu \varpi_\mu &= (-)^n \{\![H, p_a^\mu \varpi_\mu]\!\} && (41) \\ \mathbf{d} \bullet y^a \varpi_\mu &= (-)^n \{\![H, y^a \varpi_\mu]\!\}. && (42) \end{aligned}$$

Equation (32) can be understood as tantamount to the following statement

$$(-)^n \partial_\mu \langle \widehat{\mathrm{d}x^\mu \bullet} \circ \widehat{p_a^\nu \varpi_\nu} \rangle = \langle \frac{\mathrm{i}}{\hbar} [\widehat{H}, \widehat{p_a^\nu \varpi_\nu}] \rangle = \langle \widehat{\{\![H, p_a^\mu \varpi_\mu]\!\}} \rangle \qquad (43)$$

which is an Ehrenfestian version of (41), provided $\widehat{\mathrm{d}x^\mu \bullet}$ is identified with $(-)^{n-1} \varkappa \gamma^\mu$. Note that the operator $\hat{\epsilon}^\mu$ in the representation of $\hat{p}_a^\mu$ in (22) can be identified, up to a sign factor, with $\widehat{\mathrm{d}x^\mu \bullet}$. An independent evidence of that could be in principle obtained also from the consideration of geometric quantization of Poisson-Gerstenhaber brackets in the DW Hamiltonian theory (see [13]), given the fact that $\mathrm{d}x^\mu \bullet$ acts on forms similarly to the contraction with the multivector of degree $(n-1)$: $\epsilon^{\mu\mu_1 \cdots \mu_{n-1}} \partial_{\mu_a} \wedge \ldots \wedge \partial_{\mu_{n-1}}$.

Now, let us consider an Ehrenfestian version of equation (42). The operator version of the r.h.s. of (42): $\mathbf{d} \bullet y^a \varpi_\mu = \partial_\nu (\mathrm{d}x^\nu \bullet y^a \varpi_\mu)$, can be written as $\partial_\nu (\widehat{\mathrm{d}x^\nu \bullet} \circ \widehat{y^a \varpi_\mu})$. Let us consider its average

$$\begin{aligned} \partial_\nu \langle \widehat{dx^\nu \bullet} \circ \widehat{y^a \varpi_\mu} \rangle &= \partial_\nu \int \mathrm{d}y \ \mathrm{Tr}\big(\overline{\Psi} \widehat{dx^\nu \bullet} \circ \widehat{y^a \varpi_\mu} \Psi \big) \\ &= \int \mathrm{d}y \ \mathrm{Tr}\big(\partial_\nu \overline{\Psi} \widehat{\mathrm{d}x^\nu \bullet} \circ \widehat{y^a \varpi_\mu} \Psi + \overline{\Psi} \widehat{\mathrm{d}x^\nu \bullet} \circ \widehat{y^a \varpi_\mu} \partial_\nu \Psi \big) \\ &= (-)^n \mathrm{i} \int \mathrm{d}y \ \mathrm{Tr}\big(\widehat{H} \overline{\Psi} y^a \widehat{\varpi}_\mu - \overline{\Psi} y^a \widehat{\varpi}_\mu \widehat{H} \Psi \big) \\ &= (-)^n \mathrm{i} \int \mathrm{d}y \ \mathrm{Tr}\big(\overline{\Psi} [\widehat{H}, y^a \widehat{\varpi}_\mu] \Psi \big) = (-)^n \langle \widehat{\{\![H, y^a \varpi_\mu]\!\}} \rangle \end{aligned} \qquad (44)$$

where in the third line we have used the property of the composition of operators $\widehat{\mathrm{d}x^\mu \bullet}$ and $\widehat{\varpi}_\nu$: $\widehat{\mathrm{d}x^\mu \bullet} \circ \widehat{\varpi}_\nu - \widehat{\varpi}_\nu \circ \widehat{\mathrm{d}x^\mu \bullet} = 0$, which results from quantization of one of the fundamental brackets in (21), (22). Equation (44) shows that the bracket form of the second DW Hamiltonian equation (42) is also fulfilled on the average.

4. Ehrenfest Theorem in Pure Yang–Mills Theory

The Lagrangian density of pure Yang–Mills theory reads

$$L = -\frac{1}{4} F_{a\mu\nu} F^{a\mu\nu} \tag{45}$$

where

$$F^a_{\mu\nu} := \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g C^a{}_{bc} A^b_\mu A^c_\nu \tag{46}$$

g is the Yang-Mills self-coupling constant and C_{abc} are totally antisymmetric structure constants which fulfill the Jacobi identity

$$C^e{}_{ab} C^d{}_{ec} + C^e{}_{bc} C^d{}_{ea} + C^e{}_{ca} C^d{}_{eb} = 0. \tag{47}$$

The polymomenta and the DW Hamiltonian are given by

$$\pi^{\nu\mu}_a := \frac{\partial L}{\partial(\partial_\mu A^a_\nu)} = -\partial^\mu A^\nu_a + \partial^\nu A^\mu_a - g C_{abc} A^b_\mu A^c_\nu = F^{\nu\mu}_a \tag{48}$$

$$H = \pi^{\nu\mu}_a \partial_\mu A^a_\nu - L = -\frac{1}{4} \pi_{a\mu\nu} \pi^{a\mu\nu} + \frac{g}{2} C^a{}_{bc} A^b_\mu A^c_\nu \pi^{\mu\nu}_a. \tag{49}$$

The definition of polymomenta leads to the primary constraint (in the sense of the DW Hamiltonian theory[2])

$$\pi^{\mu\nu}_a + \pi^{\nu\mu}_a \approx 0. \tag{50}$$

The Yang-Mills field equations in DW Hamiltonian form read:

$$\partial_\mu \pi^{\nu\mu}_a = -\frac{\partial H}{\partial A^a_\nu} = -g\, C_{abc} A^b_\mu \pi^{\nu\mu}_c \tag{51}$$

$$\partial_{[\mu} A^a_{\nu]} \;=\; \frac{\partial H}{\partial \pi^{\nu\mu}_a} \;= \frac{1}{2} \pi^a_{\mu\nu} - \frac{1}{2} g\, C^a_{bc} A^b_\mu A^c_\nu. \tag{52}$$

The antisymmetrization in the left hand side of the second equation makes the DW Hamiltonian equations consistent with the primary constraints.

Let us note that the related treatments of classical YM theory within the multisymplectic framework can be found in [2, 8, 25]. Precanonical quantization of YM theory, its connection with the functional Schrödinger representation, and a potential application to the mass gap problem have been discussed earlier in [21].

Precanonical quantization leads to the representation of polymomenta as

$$\hat{\pi}^{\mu\nu}_a = -\mathrm{i}\hbar\varkappa\gamma^\nu \partial_{A^a_\mu}. \tag{53}$$

[2] An extension of the Dirac's theory of constraints and the Dirac bracket to the DW Hamiltonian theory has been discussed in [20].

The primary constraint (50) is taken into account as the constraint on the physical quantum states

$$\hat{\pi}_a^{(\nu\mu)}\,|\Psi\rangle^{\,\mathrm{phys}} = 0 \tag{54}$$

whence it follows $\langle\hat{\pi}_a^{(\nu\mu)}\rangle^{\mathrm{phys}} = 0$. From (49) we obtain the DW Hamiltonian operator

$$\widehat{H} = \frac{1}{2}\hbar^2\varkappa^2\frac{\partial}{\partial A_a^\mu \partial A_\mu^a} - \frac{1}{2}\mathrm{i}g\hbar\varkappa C^a{}_{bc}A_\mu^b A_\nu^c\gamma^\nu\frac{\partial}{\partial A_\mu^a}\;. \tag{55}$$

Note that in quantum YM theory the DW Hamiltonian operator is not scalar and the second term, which is responsible for self-interaction, is Clifford algebra valued.

The quantum states are represented by Clifford-valued wave functions $\Psi(A_a^\mu, x^\nu)$ with the scalar product given by

$$\langle\Phi|\Psi\rangle = \int[\mathrm{d}A]\ \mathrm{Tr}\big(\overline{\Phi}\Psi\big) \tag{56}$$

where the measure $[\mathrm{d}A] = \prod_{a,\mu}\mathrm{d}A_\mu^a$. The conservation law

$$\partial_\mu\int[\mathrm{d}A]\ \mathrm{Tr}\Big(\overline{\Psi}\gamma^\mu\Psi\Big) = 0 \tag{57}$$

follows from the precanonical Schrödinger equation (25) and its conjugate (29), and the fact that the DW Hamiltonian operator of pure YM system is generalized self-adjoint in the sense that $\hat{H} = \beta\hat{H}^\dagger\beta$, because $\beta\gamma^{\mu\dagger}\beta = \gamma^\mu$.

Now, a straightforward calculation yields

$$\begin{aligned}\partial_\nu\langle\hat{\pi}_a^{\mu\nu}\rangle &= -\,\mathrm{i}\hbar\varkappa\partial_\nu\int[\mathrm{d}A]\ \mathrm{Tr}\Big(\overline{\Psi}\gamma^\nu\partial_{A_\mu^a}\Psi\Big)\\ &= \int[\mathrm{d}A]\ \mathrm{Tr}\Big((\hat{H}\overline{\Psi})\partial_{A_\mu^a}\Psi - \overline{\Psi}\partial_{A_\mu^a}\circ\hat{H}\Psi\Big) = -\langle\partial_{A_\mu^a}\hat{H}\rangle.\end{aligned} \tag{58}$$

Therefore, the first of the YM field equations in DW Hamiltonian form, equation (51), is proven to be satisfied on the average.

The validity of the Ehrenfest theorem for the second YM field equation (52) can be proven similarly to the calculation in (44)

$$\begin{aligned}
\partial_{[\nu} A^a_{\mu]} &= (-)^n \partial_\alpha \langle (A^a_{[\mu} \mathrm{d}x^\alpha \bullet \circ \varpi_{\nu]})^{op} \rangle \\
&= (-)^n \partial_\alpha \int [\mathrm{d}A] \ \mathrm{Tr}\Big(\overline{\Psi} A^a_{[\mu} \widehat{\mathrm{d}x^\alpha \bullet} \circ \widehat{\varpi}_{\nu]} \Psi \Big) \\
&= \mathrm{i} \int [\mathrm{d}A] \ \mathrm{Tr}\Big(\hat{H} \overline{\Psi} A^a_{[\mu} \widehat{\varpi}_{\nu]} \Psi - \overline{\Psi} A^a_{[\mu} \widehat{\varpi}_{\nu]} \hat{H} \Psi \Big) \\
&= \mathrm{i} \int [\mathrm{d}A] \ \mathrm{Tr}\Big(\overline{\Psi} [\hat{H}, A^a_{[\mu} \hat{\varpi}_{\nu]}] \Psi \Big) \\
&= \int [\mathrm{d}A] \ \mathrm{Tr}\Big(\overline{\Psi} (\{\![H, A^a_{[\mu} \varpi_{\nu]}]\!\})^{op} \Psi \Big) = \Big\langle \frac{\widehat{\partial H}}{\partial \pi^{\mu\nu}_a} \Big\rangle .
\end{aligned} \tag{59}$$

Thus, we have shown that the DW Hamiltonian form of YM field equation arises as the equation for the expectation values of precanonically quantized operators.

5. Ehrenfest Theorem in Curved Space-Time

Let us consider interacting scalar fields on curved space-time background $g^{\mu\nu}(x)$. The dynamics is given by the Lagrangian density

$$\mathfrak{L} = \frac{1}{2} \sqrt{g} g^{\mu\nu} \partial_\mu y^a \partial_\nu y_a - \sqrt{g} V(y) \tag{60}$$

where $g := |\mathrm{det} g_{\mu\nu}|$, and the designation of the parametric dependence from x-s is omitted here and in what follows. In this case the polymomenta

$$\mathfrak{p}^\mu_a = \frac{\partial \mathfrak{L}}{\partial \partial_\mu y^a} = \sqrt{g} g^{\mu\nu} \partial_\mu y_a \tag{61}$$

the DW Hamiltonian density

$$\mathfrak{H} = \sqrt{g} H = \frac{1}{2\sqrt{g}} g_{\mu\nu} \mathfrak{p}^\mu_a \mathfrak{p}^{a\nu} + \sqrt{g} V(y) \tag{62}$$

and the polysymplectic structure

$$\Omega = \mathrm{d}\mathfrak{p}^\mu_a \wedge \mathrm{d}\phi^a \wedge \varpi_\mu \tag{63}$$

are densities of the weight $+1$, which parametrically depend on the space-time coordinates x. Note that in our notation the differentials d in (63) do not act on x-s, as they are "vertical" (for the mathematical details of the definition of this notion, see [14]).

The DW Hamiltonian equations of the system of scalar fields given by $\mathfrak{L}$ read

$$\partial_\mu \mathfrak{p}_a^\mu(x) = -\frac{\partial \mathfrak{H}}{\partial y^a}, \qquad \partial_\mu y^a(x) = \frac{\partial \mathfrak{H}}{\partial \mathfrak{p}_a^\mu} \tag{64}$$

where ∂_μ acts both on the parametric dependence on x via $g^{\mu\nu}(x)$ and the dependence on x due to the pull back to a specific section in the polymomentum phase space of variables $(\mathfrak{p}_a^\mu, y^a)$, which represents a solution of classical field equations. Note that we could obtain the same equations by applying the usual rules of covariantization to the DW equations in flat space-time.

The Poisson bracket operation defined by the weight $+1$ density valued polysymplectic structure (63) has a density weight -1, so that, for example,

$$\{\![\mathfrak{p}_a^\mu(x), y^b \varpi_\nu]\!\} = \delta_a^b \delta_\nu^\mu. \tag{65}$$

The Dirac quantization rule in curved space-time is also modified to make sure that density valued quantities are quantized as density valued operators of the same weight

$$[\hat{A}, \hat{B}] = -\mathrm{i}\hbar\sqrt{g}\,\widehat{\{\![A, B]\!\}}. \tag{66}$$

It leads to the following representations

$$\hat{\mathfrak{p}}_a^\mu = -\mathrm{i}\hbar\varkappa\sqrt{g}\gamma^\mu\partial_a, \qquad \widehat{H} = -\frac{1}{2}\hbar^2\varkappa^2\partial_a\partial^a + V(y) \tag{67}$$

where the x-dependent γ-matrices are introduced such that $\gamma^\mu\gamma^\mu + \gamma^\mu\gamma^\mu = 2g^{\mu\nu}$. Note that the operator of the DW Hamiltonian does not contain x-dependent quantities.

The curved space-time version of the precanonical Schrödinger equation (25) takes the form

$$\mathrm{i}\hbar\varkappa\gamma^\mu(x)\nabla_\mu\Psi = \widehat{H}\Psi \tag{68}$$

where $\nabla_\mu := \partial_\mu + \omega_\mu(x)$ is a covariant derivative of Clifford algebra valued wave functions. Let us see if the requirement that the Ehrenfest theorem extends also to the case of curved space-time can help us to specify the connection term $\omega_\mu(x)$.

Before we proceed, let us find the precanonical Schrödinger equation for the conjugate wave function $\overline{\Psi} := \bar{\beta}\Psi^\dagger\bar{\beta}$, where $\bar{\gamma}^I$, $I = 0, ..., n-1$ denote the flat (tangent) space Dirac matrices, such that $\bar{\gamma}^I\bar{\gamma}^J + \bar{\gamma}^J\bar{\gamma}^I = 2\eta^{IJ}$, η^{IJ} is the Minkowski metric, and $\bar{\beta} := \bar{\gamma}^0$. If $\hat{H}$ is generalized self-adjoint: $\hat{H} = \bar{\beta}\hat{H}^\dagger\bar{\beta}$, by multiplying the Hermite conjugate of (68) by $\bar{\beta}$ from the left and right, and inserting $\bar{\beta}^2 = 1$ where needed, we obtain

$$\mathrm{i}\hbar\varkappa\overline{\Psi}(\overleftarrow{\partial}_\mu + \overline{\omega}_\mu)\gamma^\mu = -\hat{H}\overline{\Psi} \tag{69}$$

where $\overline{\omega}_\mu := \bar{\beta}\omega_\mu^\dagger\bar{\beta}$ (not to be confused with ϖ_μ in (63)!).
Let us consider a conservation law, which would generalize (31) to curved space-time

$$\begin{aligned} \mathrm{i}\partial_\mu \int \mathrm{d}y \ \mathrm{Tr}\Big(\overline{\Psi}\sqrt{g}\gamma^\mu\Psi\Big) &= \mathrm{i}\int \mathrm{d}y \ \mathrm{Tr}\Big(\partial_\mu\overline{\Psi}\sqrt{g}\gamma^\mu\Psi + \overline{\Psi}\sqrt{g}\gamma^\mu\partial_\mu\Psi \\ &\qquad + \overline{\Psi}\partial_\mu(\sqrt{g}\gamma^\mu)\Psi\Big) \qquad (70) \\ &= \int \mathrm{d}y \ \mathrm{Tr}\Big(\overline{\Psi}\sqrt{g}\big(-\hat{H} - \mathrm{i}\overline{\omega}_\mu\gamma^\mu\big)\Psi + \overline{\Psi}\sqrt{g}\big(\hat{H} - \mathrm{i}\gamma^\mu\omega_\mu\big)\Psi + \overline{\Psi}\mathrm{i}\partial_\mu(\sqrt{g}\gamma^\mu)\Psi\Big) \\ &= \int \mathrm{d}y \ \mathrm{Tr}\Big(\overline{\Psi}\mathrm{i}\big(-\sqrt{g}\overline{\omega}_\mu\gamma^\mu - \sqrt{g}\gamma^\mu\omega_\mu + \partial_\mu(\sqrt{g}\gamma^\mu)\big)\Psi\Big). \end{aligned}$$

Therefore, the covariant version of the conservation law (31)

$$\partial_\mu \int \mathrm{d}y \ \mathrm{Tr}\Big(\overline{\Psi}\sqrt{g}\gamma^\mu\Psi\Big) = 0$$

is fulfilled if the connection ω_μ satisfies the equality

$$\sqrt{g}\overline{\omega}_\mu\gamma^\mu + \sqrt{g}\gamma^\mu\omega_\mu - \partial_\mu(\sqrt{g}\gamma^\mu) = 0. \qquad (71)$$

Furthermore,

$$\begin{aligned} \mathrm{i}\partial_\mu \int \mathrm{d}y \ \mathrm{Tr}\Big(\overline{\Psi}\sqrt{g}\gamma^\mu\partial_a\Psi\Big) &= \int \mathrm{d}y \ \mathrm{Tr}\Big(\mathrm{i}\partial_\mu\overline{\Psi}\sqrt{g}\gamma^\mu\partial_a\Psi \\ &\quad + \overline{\Psi}\partial_a\sqrt{g}\gamma^\mu\mathrm{i}\partial_\mu\Psi + \overline{\Psi}\mathrm{i}\partial_\mu(\sqrt{g}\gamma^\mu)\Psi\Big). \end{aligned} \qquad (72)$$

By substituting the precanonical Schrödinger equation in curved space-time and its conjugate we obtain in the r.h.s. of (72)

$$\begin{aligned} \int \mathrm{d}y \ \mathrm{Tr}\Big(&-\sqrt{g}\hat{H}\overline{\Psi}\partial_a\Psi - \mathrm{i}\overline{\Psi}\sqrt{g}\overline{\omega}_\mu\gamma^\mu\partial_a\Psi \\ &+ \overline{\Psi}\partial_a \circ \sqrt{g}\big(\hat{H} - \mathrm{i}\gamma^\mu\omega_\mu\big)\Psi + \mathrm{i}\overline{\Psi}\partial_\mu(\sqrt{g}\gamma^\mu)\partial_a\Psi\Big). \end{aligned} \qquad (73)$$

The terms with $\hat{H}$ yield

$$\begin{aligned} \int \mathrm{d}y \ \mathrm{Tr}\Big(&-\overline{\Psi}\sqrt{g}\hat{H}\circ\partial_a\Psi + \overline{\Psi}\partial_a\circ\sqrt{g}\hat{H}\Psi\Big) \\ &= \int \mathrm{d}y \ \mathrm{Tr}\Big(-\overline{\Psi}(\partial_a\hat{\mathfrak{H}})\Psi\Big) = -\langle\partial_a\hat{\mathfrak{H}}\rangle. \end{aligned} \qquad (74)$$

Therefore, the first DW Hamiltonian equation in (64) is fulfilled on the average if the remaining three terms in (73)

$$\int \mathrm{d}y \ \mathrm{Tr}\Big(\overline{\Psi}\big(- \sqrt{g}\varpi_\mu \gamma^\mu - \sqrt{g}\gamma^\mu \omega_\mu + \partial_\mu (\sqrt{g}\gamma^\mu)\big)\partial_a \Psi\Big) \tag{75}$$

produce a vanishing result. This condition limits the choice of the connection ω_μ and it coincides with (71).

Now, let us consider the covariant version of equation (38)

$$\nabla_\mu (y\varpi^\mu) = \partial_\mu (y^a \varpi^\mu) + \frac{1}{2} y \partial_\mu (\ln g) \varpi^\mu . \tag{76}$$

Let us see if we can obtain it on the average from the precanonical Schrödinger equation on curved space-time. By a straightforward calculation we obtain

$$\begin{aligned}
\mathrm{i}\partial_\mu \langle \widehat{y^a \varpi^\mu} \rangle &= \mathrm{i} \int \mathrm{d}y \ \mathrm{Tr}\Big(\partial_\mu \overline{\Psi} y^a \gamma^\mu \Psi + \overline{\Psi} y^a \gamma^\mu \partial_\mu \Psi + \overline{\Psi} y^a (\partial_\mu \gamma^\mu)\Psi\Big) \\
&= \int \mathrm{d}y \ \mathrm{Tr}\Big(\overline{\Psi}\big(- \overleftarrow{\hat{H}} - \mathrm{i}\varpi_\mu \gamma^\mu\big) y^a \Psi + \overline{\Psi} y^a \big(\hat{H} - \mathrm{i}\gamma^\mu \omega_\mu\big)\Psi + \overline{\Psi} y^a (\mathrm{i}\partial_\mu \gamma^\mu)\Psi\Big) \\
&= \int \mathrm{d}y \ \mathrm{Tr}\Big(\overline{\Psi}[y^a, \hat{H}]\Psi + \mathrm{i}\overline{\Psi}\big(- \varpi_\mu \gamma^\mu - \gamma^\mu \omega_\mu + \partial_\mu \gamma^\mu\big) y^a \Psi\Big).
\end{aligned} \tag{77}$$

Therefore, equation (76) and the second DW Hamiltonian equation in (64) are fulfilled on the average if the connection ω_μ satisfies the condition

$$\varpi_\mu \gamma^\mu + \gamma^\mu \omega_\mu - \partial_\mu \gamma^\mu = \frac{1}{2} \partial_\mu (\ln g) \gamma^\mu \tag{78}$$

which is again equivalent to the condition obtained in (71).

One can view equation (71) for the connection term as a consequence of a covariant constancy of the curved space-time Dirac matrices $\gamma^\mu(x)$ or, equivalently, the vielbeins $e^\mu_I(x)$. This is what identifies the connection term ω_μ in (68) with the spin-connection: $\omega_\mu = \omega_\mu^{IJ} \bar{\gamma}_{IJ} = -\varpi_\mu$ with real coefficients ω_μ^{IJ}. As the Clifford-valued precanonical wave function can be also viewed as a spinor field with two spinor indices originating from the indices of γ-matrices, the appearance of the spin connection in the Dirac operator in (68) is natural here (see e.g. [28] for a related discussion).

6. Conclusions

We have shown that in the context of precanonical quantization of fields the evolution (or rather, space-time variation) of expectation values of fundamental operators is consistent with classical field equations in DW Hamiltonian form. This

property can be considered as a consistency test of three different aspects of precanonical quantization playing together: the precanonical representation of quantum operators in terms of Clifford-valued operators, the precanonical Schrödinger equation in (25), and the scalar product for the calculation of expectation values of operators using the Clifford-valued precanonical wave functions in (34).

We have explicitly demonstrated that the Ehrenfest theorem can be proven for the system of interacting scalar fields both in flat and curved space-time, and for precanonically quantized pure Yang-Mills theory. In curved space-time the consideration of the Ehrenfest theorem leads to the condition on the admissible connection term in the Dirac operator in the precanonical Schrödinger equation, which is compatible with the known properties of the spin-connection.

In our recent papers we have considered an application of precanonical quantization to the problem of quantization of gravity both in metric [22] and in vielbein [23] variables. We hope that it will be possible to demonstrate that the Einstein equations are also satisfied on the average as a consequence of our precanonical Schrödinger equation for quantum gravity, the precanonical representation of quantum operators appearing in our formulation, and the definition of the analogue of the Hilbert space of the theory which, in vielbein formulation [23], involves an operator-valued measure on the space of spin-connection coefficients.

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